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On the Limit Distributions of the Zeros of Jonquière Polynomials and Generalized Classical Orthogonal Polynomials

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Jonquière polynomials J_k are defined by the rational function $\sum_0^\infty n^k z^n = J_k(z)/(1-z)^{k+1}$, $k \in \mathbb{N}_0$. For a general class of polynomials including J_k, the limit distribution of its zeros is computed. Recently Dette and Studden have found the asymptotic zero distributions for Jacobi, Laguerre, and Hermite polynomials $P_n^{(\alpha_n, \beta_n)}$, $L_n^{(\alpha_n)}$, and $H_n^{(\alpha_n)}$ with degree dependent parameters α_n, β_n by using a continued fraction technique. In this paper these limit distributions are derived via a differential equation approach.

0. INTRODUCTION AND SUMMARY

In this paper we are concerned with some sequences of polynomials, the zeros of which are real, and in particular we deal with the computation of the limit distributions of their zeros.

First we consider Jonquière polynomials, J_k say, which may be defined by the power series expansion of the rational function

$$\sum_{n=0}^{\infty} n^k z^n = \left(z\frac{d}{dz}\right)^k \frac{1}{1-z} =: \frac{J_k(z)}{(1-z)^{k+1}}, \qquad k \in \mathbb{N}_0, \tag{0.1}$$

[cf. 27, problem 46, p. 7; 16]. Sometimes, apparently in numerical analysis, J_k and its modifications also are called Euler–Frobenius polynomials. Obviously J_k is a polynomial of degree k, and by Rolle's theorem all its zeros are simple, real, and nonpositive. Jonquière polynomials and their zeros play an important role in various parts of mathematics and its applications, for instance in summability theory [25, Chap. IV.3], approximation theory [11, 18, 23, 28], and in the theory of the structure of polymers [30]. The zeros in particular have been investigated in great detail; some of the corresponding work in the literature is concerned with

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inequalities [e.g. 11], parameter dependence [11, 34], and asymptotic properties [6, 10, 28] as well. Denoting the zeros of J_k by $x_{k\nu}$, which we may assume to be numbered according to

$$x_{k1} < x_{k2} < \cdots < x_{kk}, \qquad k \in \mathbb{N}, \tag{0.2}$$

and denoting its counting function by

$$N_k(\xi) := |\{\nu \in \{1, ..., k\} \mid x_{k\nu} \leqslant \xi\}|, \qquad \xi \in \mathbb{R}, \tag{0.3}$$

the global asymptotic behaviour of the zeros has been described by the limit distribution [6, 10]

$$\lim_{k \to \infty} \frac{1}{k} N_k(\xi) = \int_{-\infty}^{\xi} \frac{dt}{(-t)(\log^2(-t) + \pi^2)}, \qquad -\infty < \xi \leqslant 0, \tag{0.4}$$

which may be regarded as a "logarithmic" Cauchy distribution. Generalizing Jonquière's function (0.1) in [9, 15], power series of the type

$$f_k(z) = \sum_{n=0}^{\infty} E_k(n)\, z^n \tag{0.5}$$

have been investigated with regard to various properties of their zeros. Here E_k denotes an exponential polynomial that is

$$E_k(x) = \sum_{\nu=1}^{p} P_{k_\nu - 1}(x)\, e^{\alpha_\nu x}, \qquad x \in \mathbb{C}, \tag{0.6}$$

say, $P_{k_\nu - 1}$ being polynomials and $\alpha_\nu \in \mathbb{C}$. Then clearly (0.5) defines a rational function as (0.1) does. For our purpose it is convenient to describe E_k as a solution of a linear differential equation with constant coefficients. To this end we suppose throughout that Q_{k+1} is a real polynomial with (exact) degree $k+1 \geqslant 2$ and zeros $\beta_{k+1,\nu}$, $\nu = 1, ..., k+1$. If E_k is a non-identically vanishing and real solution of the initial value problem

$$Q_{k+1}\left(\frac{d}{dx}\right) E_k(x) = 0, \qquad E_k^{(\nu)}(0) = 0, \quad \nu = 0, ..., k-1, \tag{0.7}$$

(which is uniquely determined except for a multiplicative constant; see Lemma 1) and if $|\operatorname{Im} \beta_{k+1,\nu}| < \pi$, $\nu = 1, ..., k+1$, then among other results in [9] it is proved that f_k has exactly k zeros in the plane which are simple ones and all of them are located on the nonpositive real axis. Under these conditions in particular we may rewrite (0.5) as (see Section 1)

$$f_k(z) = \frac{J_k(z)}{\prod_{\nu=1}^{k+1} (1 - z e^{\beta_{k+1,\nu}})}, \tag{0.5'}$$

where now as in the special case (0.1) (that is, $Q_{k+1}(x)=x^{k+1}$) we denote the numerator polynomial in (0.5′) by J_k again and call it a Jonquière polynomial, too. Of course, J_k in (0.5′) depends on Q_{k+1}. In continuation of the limiting result (0.4) for the particular case (0.1) we ask for a possible asymptotic distribution function for the zeros of J_k in (0.5′), provided Q_{k+1} in (0.7) possesses a "nice asymptotic behaviour."

Attributing equal weights $1/(k+1)$ to each of the zeros $\beta_{k+1,\nu}$ of Q_{k+1}, we may rewrite the distribution function of these zeros as

$$\mu_k(t) := \frac{1}{k+1} \sum_{\nu=1}^{k+1} \delta_{\beta_{k+1,\nu}}(t), \qquad t \in \mathbb{C}, \tag{0.8}$$

where $\delta_\beta(t)$ denotes the unit mass function at $t=\beta$. Now the central result of this paper reads as follows. If μ_k converges to some probability measure, μ say, on the plane, then under some natural additional assumptions and with the notations of (0.2) and (0.3) we have

$$\lim_{k\to\infty} \frac{1}{k} N_k(\xi) = \int_{-\infty}^{\xi} g(x)\,dx, \qquad -\infty < \xi \leqslant 0, \tag{0.9}$$

where

$$g(x) := \frac{-1}{\pi x} \int \frac{\pi - \operatorname{Im} t}{(\log(-1/x) - \operatorname{Re} t)^2 + (\pi - \operatorname{Im} t)^2}\, d\mu(t), \qquad -\infty < x < 0, \tag{0.10}$$

the integral being taken over the strip $|\operatorname{Im} t| < \pi$ (Theorem 1).

Formula (0.4) corresponds to the case $\mu_k = \mu = \delta_0$. Other examples and special cases are also treated in Section 1. As in [6–8] our proofs are based on a theorem of Grommer and Hamburger which may be looked at as a continuity theorem for the Stieltjes transform of probability distributions.

In Section 2 we consider Jacobi, Laguerre, and Hermite polynomials which throughout we denote by $P_n^{(\alpha,\beta)}$, $L_n^{(\alpha)}$, and $H_n^{(\gamma)}$ respectively, where the real numbers α, β, γ are chosen subject to $\alpha, \beta > -1$ and $\gamma > -1/2$. The definitions of these classical orthogonal polynomials and some well known formulae as well we take from Szegö's classic memoir [29, Chap. V] and Chihara's book [2, p. 157]. In particular, we only mention that these orthogonal polynomials on the real line pertain to the weight functions $(1-x)^\alpha (1+x)^\beta$ on $(-1, 1)$, $x^\alpha e^{-x}$ on $(0, \infty)$, and $|x|^{2\gamma} e^{-x^2}$ on $(-\infty, \infty)$ respectively. Further, all its zeros are contained in the cited intervals under the parameter restrictions given above. Various questions of constrained or weighted polynomial approximations are closely related to these classical orthogonal polynomials; however, now with degree dependent weights the parameters $\alpha = \alpha_n$, $\beta = \beta_n$, and $\gamma = \gamma_n$ may depend on the degree n.

Sharpening a theorem of Moak *et al.* [22] on the denseness of the zeros of Jacobi polynomials $P_n^{(\alpha_n, \beta_n)}$ on certain subintervals of $[-1, 1]$, Dette and Studden [4] derived the asymptotic distribution functions of the zeros for each of the polynomials $P_n^{(\alpha_n, \beta_n)}$, $L_n^{(\alpha_n)}$, and $H_n^{(\gamma_n)}$, provided that α_n, $\beta_n > -1$, $\gamma_n > -1/2$, and they satisfy certain natural limiting conditions. Their proofs essentially rely on a characterization theorem for these polynomial sequences along with a continued fraction technique [3]. The second object of the present paper is to give an alternative computation of the limit distributions for the zeros of $P_n^{(\alpha_n, \beta_n)}$, $L_n^{(\alpha_n)}$, and $H_n^{(\gamma_n)}$. In contrast to [4] our proof is based on the characterizing differential equations (see Lemma 3) and the above mentioned continuity theorem for the Stieljes transform of probability distributions [see also 6–8].

1. Jonquière Polynomials

In the following we prove the main result of this paper. To this end we first collect some auxiliary results which are basic for the technical treatments of its proof. We consider exponential polynomials (0.6) as solutions of a differential equation. Introducing the polynomial

$$Q_{k+1}(x) := \prod_{\nu=1}^{k+1} (x - \beta_{k+1,\nu}), \tag{1.1}$$

where $k \in \mathbb{N}_0$ and $\beta_{k+1,\nu} \in \mathbb{C}$, $\nu = 1, \ldots, k+1$, the following lemma is shown by straightforward arguments [cf. 26, p. 356], so that we can omit the proof.

LEMMA 1. *Suppose that $k \in \mathbb{N}_0$. Then the solution E_k of the initial value problem*

$$Q_{k+1}\left(\frac{d}{dx}\right) E_k(x) = 0, \qquad E_k^{(\nu)}(0) = 0, \quad \nu = 0, \ldots, k-1, \qquad E_k^{(k)}(0) = c \neq 0 \tag{1.2}$$

can be written as

$$E_k(x) = \frac{c}{2\pi i} \int_C \frac{e^{\zeta x}}{Q_{k+1}(\zeta)}\, d\zeta, \qquad x \in \mathbb{C},$$

C being a simple closed contour with positive orientation enclosing all the zeros of Q_{k+1}.

Next, we derive explicit representations for the analytic extension of Jonquière's function f_k under the condition (1.2). If Q_{k+1} is given by (1.1) and $\alpha_{k+1,1},..., \alpha_{k+1,p}$ denote the distinct roots among $\beta_{k+1,1},...,\beta_{k+1,k+1}$, then in the following we use the abbreviation

$$\mathbb{C}_k^* := \mathbb{C} \setminus \{e^{-\alpha_{k+1,1}}, ..., e^{-\alpha_{k+1,p}}\}.$$

LEMMA 2. *Assume that $k \in \mathbb{N}$ and Q_{k+1} is given by* (1.1). *If E_k is the solution of the initial value problem* (1.2), *then the power series $f_k(z) := \sum_{n=0}^{\infty} E_k(n) z^n$ possesses a unique analytic extension onto $\mathbb{C}_k^*$ with representations*

$$f_k(z) = \frac{P_k(z)}{\prod_{\nu=1}^{k+1} (1 - ze^{\beta_{k+1,\nu}})}, \tag{1.3}$$

P_k being a polynomial of degree $\leqslant k$ and

$$f_k(z) = c \sum_{m=-\infty}^{\infty} \frac{1}{Q_{k+1}(2\pi im + \log(1/z))}. \tag{1.4}$$

The series in (1.4) *is compactly convergent on $\mathbb{C}_k^*$ and* $\log(1/z)$ *is the principal branch meaning* $\log(1/z)$ *is real for positive z.*

In case $Q_{k+1}(x) = x^{k+1}$ the representation (1.4) sometimes is called the Lindelöf–Wirtinger expansion [cf. 17, 35].

Proof. Employing Lemma 1 for sufficiently small $|z|$ we obtain

$$f_k(z) = \frac{c}{2\pi i} \int_C \frac{1}{Q_{k+1}(\zeta)} \frac{d\zeta}{1 - ze^{\zeta}}, \tag{1.5}$$

C being chosen such that all the poles of $1/Q_{k+1}$ are located in the interior of C and all the poles $\zeta_m := 2\pi im + \log(1/z)$, $m \in \mathbb{Z}$, of $1/(1 - ze^{\zeta})$ in the exterior of C. A direct application of the residue theorem produces (1.3). To verify (1.4) it is easily seen that there exists a sequence of circles, C_{R_ν} say, with center at the origin and radius R_ν satisfying $R_\nu \to \infty$ as $\nu \to \infty$ and

$$\operatorname{dist}(C_{R_\nu}, \zeta_m) \geqslant \delta \qquad \text{for all} \quad \nu \in \mathbb{N}, \quad m \in \mathbb{Z},$$

where δ is some positive constant. Then the quantity $\sup_{\zeta \in C_{R_\nu}} |1 - ze^{\zeta}|^{-1}$ is finite and does not depend on ν. Now another routine application of residue calculus to (1.5) leads to the expansion (1.4). ∎

In the remainder of the paper we suppose the polynomial (1.1) to be *real* with zeros $\beta_{k+1,\nu} \in \mathbb{C}$ satisfying

$$|\operatorname{Im} \beta_{k+1,\nu}| < \pi, \qquad \nu = 1, ..., k+1, \tag{1.6}$$

and that E_k is a *real* solution of the initial value problem (1.2) where $k \in \mathbb{N}$. Then, by Lemma 2, the power series (0.5) $f_k(z) := \sum_{n=0}^{\infty} E_k(n) z^n$ defines a rational function, that is

$$f_k(z) = \frac{J_k(z)}{\prod_{\nu=1}^{k+1} (1 - ze^{\beta_{k+1,\nu}})}, \tag{0.5'}$$

J_k being the Jonquière polynomial associated with $\beta_{k+1,\nu}$, $\nu = 1, ..., k+1$. Moreover, by [9, Theorem 1, p. 263], J_k has exactly k zeros and all of them are real, nonpositive, and simple (observe (1.6)). If the distribution functions $(1/k)\, N_k(\xi)$ and $\mu_k(t)$ of the zeros of J_k and Q_{k+1} are defined by (0.3) and (0.8), respectively, then we prove

THEOREM 1. *With the notations and assumptions above suppose that the set $K \subset \{t \mid |\mathrm{Im}\, t| < \pi\}$ is compact such that*

$$\beta_{k+1,\nu} \in K, \qquad \nu = 1, ..., k+1, \quad k \in \mathbb{N},$$

and μ_k converges to a probability measure μ on the plane. Then

$$\lim_{k \to \infty} \frac{1}{k} N_k(\xi) = \int_{-\infty}^{\xi} g(x)\, dx, \qquad -\infty < \xi \leqslant 0, \tag{0.9}$$

where

$$g(x) := \frac{-1}{\pi x} \int_K \frac{\pi - \mathrm{Im}\, t}{(\log(-1/x) - \mathrm{Re}\, t)^2 + (\pi - \mathrm{Im}\, t)^2}\, d\mu(t), \qquad -\infty < x < 0. \tag{0.10}$$

Proof. First we note that all measures μ_k and μ are supported by K, which obviously we may assume to be symmetric with respect to the real axis. For results on weak convergence of probability measures on $\mathbb{C}$ we refer the reader to [1, Sect. 29]. By Lemma 2 (put $c := 1$), we have

$$J_k(z) = \prod_{\nu=1}^{k+1} (1 - ze^{\beta_{k+1,\nu}}) \cdot \sum_{m=-\infty}^{\infty} \frac{1}{Q_{k+1}(2\pi i m + \log(1/z))},$$

where for $z \in \mathbb{C}_- := \{z \in \mathbb{C} \mid |\arg z| < \pi\}$ the logarithm is defined by $\log(1/z) := \log(1/|z|) - i \arg z$. Next, by taking logarithms we obtain

$$\begin{aligned}\log J_k(z) = {} & \sum_{\nu=1}^{k+1} \log(1 - ze^{\beta_{k+1,\nu}}) - \log Q_{k+1}(\log(1/z)) \\ & + \log\left\{1 + \sum_{m \neq 0} \frac{Q_{k+1}(\log(1/z))}{Q_{k+1}(2\pi i m + \log(1/z))}\right\},\end{aligned}$$

and further differentiation yields

$$\frac{1}{k+1}\frac{J_k'(z)}{J_k(z)}=\int_K g_z(t)\,d\mu_k(t)+R_k(z) \tag{1.7}$$

where

$$g_z(t):=\frac{1}{z(\log(1/z)-t)}-\frac{e^t}{1-ze^t} \tag{1.8}$$

and

$$R_k(z):=\frac{1}{k+1}\frac{d}{dz}\log\left\{1+\sum_{m\neq 0}\frac{Q_{k+1}(\log(1/z))}{Q_{k+1}(2\pi im+\log(1/z))}\right\}. \tag{1.9}$$

Here for fixed $z\in\mathbb{C}_-$, g_z has a removable singularity at $t=\log(1/z)$ and thus is continuous and bounded on $\{t \mid |\mathrm{Im}\,t|<\pi\}$. Moreover, the series in (1.9) converges compactly on $\mathbb{C}_-$. Next, we show that this series tends to zero compactly on a neighbourhood of some positive z_0 as $k\to\infty$.

Since K is compact, there are a positive number δ and a real number $w_0\in\mathbb{C}\backslash K$ such that $|\mathrm{Im}\,t|\leqslant\pi-\delta$ for all $t\in K$ and $U_\delta(w_0):=\{w \mid |w-w_0|<\delta\}\subset\mathbb{C}\backslash K$. Let $w:=\log(1/z)\in U_{\delta/2}(w_0)$, then clearly we have $Q_{k+1}(2\pi im+w)\neq 0$ for all integers m. Because

$$Q_{k+1}(2\pi im+w)=\exp\left\{(k+1)\int_K\log(2\pi im+w-t)\,d\mu_k(t)\right\},\quad w\in U_{\delta/2}(w_0),$$

we have

$$\begin{aligned}\left|\frac{Q_{k+1}(w)}{Q_{k+1}(2\pi im+w)}\right|&=\exp\left\{(k+1)\int_K\log\left|\frac{w-t}{2\pi im+w-t}\right|d\mu_k(t)\right\}\\&\leqslant\left(\frac{M^2}{M^2+2\pi\delta m^2}\right)^{(k+1)/2},\end{aligned} \tag{1.10}$$

for $w\in U_{\delta/2}(w_0)$, $m\in\mathbb{Z}$, since

$$\begin{aligned}2\log\left|\frac{w-t}{2\pi im+w-t}\right|&=\log\frac{|w-t|^2}{|w-t|^2+4\pi^2m^2+4\pi|m|\,\mathrm{Im}(w-t)}\\&\leqslant\log\frac{M^2}{M^2+2\pi\delta m^2}\end{aligned}$$

for some positive constant M. Here we have used the compactness of K and the reality of w_0. Hence the estimate (1.10) implies that $R_k(z)\to 0$ as $k\to\infty$ compactly on some neighbourhood of $z_0 := e^{-w_0}$.

Now from (1.7) and the weak convergence $\mu_k \to \mu$ we obtain

$$\lim_{k \to \infty} \frac{1}{k} \frac{J_k'(z)}{J_k(z)} = h(z) \qquad \text{compactly for} \quad z \in \mathbb{C}_{-}, \tag{1.11}$$

by an application of Vitali's theorem, where

$$h(z) := \int_K g_z(t)\, d\mu(t), \qquad z \in \mathbb{C}_{-}.$$

By (1.8), we get

$$zh(z) \to \int_K d\mu(t) = 1 \qquad \text{as} \quad z \to \infty, \quad |\arg z| \leqslant \pi - \varepsilon,$$

and thus, by a theorem of Grommer and Hamburger [cf. 33, p. 104–105; 31, p. 175; see also 8] and (1.11), h is the Stieltjes transform of the limit distribution of the zeros of J_k. Finally, we conclude (0.9) and (0.10) by the Stieltjes inversion formula

$$g(x) = \frac{1}{\pi} \operatorname{Im} h(x - i0), \qquad x < 0. \quad \blacksquare$$

Remarks. Using the symmetry of the measure $\mu(t)$ with respect to the real axis in the t-plane which is a consequence of Q_{k+1} being a real polynomial, the following representation of the density g in (0.10) is readily verified:

$$g(x) = -\frac{1}{x} \int_K \frac{1}{(\log(-1/x) - t)^2 + \pi^2}\, d\mu(t), \qquad -\infty < x < 0. \tag{0.10$'$}$$

To illustrate our general result we mention some

EXAMPLES AND SPECIAL CASES. (i) If $Q_{k+1}(x) = x^{k+1}$, that is, $\mu_k = \mu = \delta_0$, then we get the basic case (0.1) with (0.4) that was mentioned in the Introduction.

(ii) Suppose that $\alpha_1, ..., \alpha_p$ are distinct complex numbers and $m_{k\nu} \in \mathbb{N}$, satisfying $\sum_{\nu=1}^{p} m_{k\nu} = k+1$, such that $Q_{k+1}(x) = \prod_{\nu=1}^{p} (x - \alpha_\nu)^{m_{k\nu}}$ is real-valued. If $\lim_{k \to \infty} m_{k\nu}/(k+1) = \lambda_\nu$, $\nu = 1, ..., p$, then we have

$$\mu_k = \sum_{\nu=1}^{p} \frac{m_{k\nu}}{k+1} \delta_{\alpha_\nu} \to \sum_{\nu=1}^{p} \lambda_\nu \delta_{\alpha_\nu} =: \mu$$

weakly as $k \to \infty$, and the density g of the limit distribution of the zeros is given by (use (0.10'))

$$g(x) = -\frac{1}{x} \sum_{v=1}^{p} \frac{\lambda_v}{(\log(-1/x) - \alpha_v)^2 + \pi^2}, \qquad x < 0.$$

In particular, this case entails the example (0.5) for which $E_k(x) = \int_0^x e^{-t} t^{k-1}\, dt$ is the incomplete Γ-function [cf. 15, p. 219]. Here we have $p = 2$, $\alpha_1 = -1$, $\alpha_2 = 0$, $m_{k1} = k$, $m_{k2} = 1$, $\lambda_1 = 1$, $\lambda_2 = 0$, and

$$g(x) = -\frac{1}{x} \frac{1}{(\log(-x) - 1)^2 + \pi^2}, \qquad x < 0.$$

(iii) Let $Q_{k+1} := P_{k+1}^{(\alpha, \beta)}$ be the Jacobi polynomial of degree $k+1$ with $\alpha, \beta > -1$ or some other sequence of orthogonal polynomials on $[-1, 1]$ belonging to the Szegö class [cf. 29, 31]; then it is well known that μ is the arcsin measure on $[-1, 1]$. Hence (0.10) reduces to

$$g(x) = -\frac{1}{x} \int_{-1}^{1} \frac{1}{(\log(-1/x) - t)^2 + \pi^2} \frac{dt}{\pi \sqrt{1 - t^2}}, \qquad x < 0.$$

Observing that μ has the Stieltjes transform

$$\frac{1}{\sqrt{\zeta^2 - 1}} = \int_{-1}^{+1} \frac{1}{\zeta - t} \frac{dt}{\pi \sqrt{1 - t^2}}, \qquad \zeta \in \mathbb{C} \backslash [-1, 1],$$

where the square root is such that $\sqrt{\zeta^2 - 1}$ is positive if $\zeta > 1$ and continuous throughout $\mathbb{C} \backslash [-1, 1]$, a straightforward computation yields for $x < 0$ and $y := \log(-1/x)$

$$\begin{aligned} g(x) &= \frac{1}{\pi x} \operatorname{Im} \frac{1}{\sqrt{(y + i\pi)^2 - 1}} \\ &= \frac{-1}{\sqrt{2}\, \pi x} \left(\frac{\pi^2 + 1 - y^2 + \sqrt{(\pi^2 + (y+1)^2)(\pi^2 + (y-1)^2)}}{(\pi^2 + (y+1)^2)(\pi^2 + (y-1)^2)} \right)^{1/2} \end{aligned}$$

(iv) Finally, we mention that by a slight modification of our methods we are able to treat power series having only zeros which are located on both the positive and negative real axes. A typical example in this context is given by

$$f_{2k+1}(z) := \sum_{n=0}^{\infty} n^k \sin \alpha n\, z^n, \qquad 0 < \alpha < \pi, \quad k \in \mathbb{N},$$

having precisely $2k+1$ zeros in $\mathbb{C}$, k of them being positive and k negative [9, p. 265]. Since

$$f_{2k+1}(z)=\frac{1}{2i}\left(\sum_{n=0}^{\infty} n^k(e^{i\alpha}z)^n-\sum_{n=0}^{\infty} n^k(e^{-i\alpha}z)^n\right),$$

from Lemma 2 we obtain

$$f_{2k+1}(z)=\frac{k!}{2i}\sum_{m=-\infty}^{\infty}\left(\frac{1}{(2\pi im-i\alpha+\log(1/z))^{k+1}}\right.$$
$$\left.-\frac{1}{(2\pi im+i\alpha+\log(1/z))^{k+1}}\right)$$

for $z\in\mathbb{C}\backslash\{e^{i\alpha}, e^{-i\alpha}\}$. Now arguments very similar to those in the proof of Theorem 1 lead to

$$\lim_{k\to\infty}\frac{1}{2k+1}N_{2k+1}(\xi)=\int_{-\infty}^{\xi} g_\alpha(x)\,dx, \qquad \xi\in\mathbb{R},$$

where

$$g_\alpha(x):=\frac{1}{2\pi x}\begin{cases}\dfrac{\alpha}{\log^2 x+\alpha^2}, & x>0\\[2mm] \dfrac{\alpha-\pi}{\log^2(-x)+(\pi-\alpha)^2}, & x<0.\end{cases}$$

2. ORTHOGONAL POLYNOMIALS

In this section we will obtain new and simplified proofs of known results for the classical orthogonal polynomials [cf. 3, 4, 7, 8, 12, 13, 19–21]. For this purpose we derive in the following lemma the limit distribution of the zeros for a sequence of polynomials satisfying a linear differential equation of a certain type. Actually, these polynomials essentially are of hypergeometric type [cf. 2, p. 150].

LEMMA 3. *Suppose that* $(a_2^{(n)})$, $(b_2^{(n)})$, $(c_2^{(n)})$, $(a_1^{(n)})$, $(b_1^{(n)})$, *and* $(a_0^{(n)})$ *are sequences of real numbers satisfying*

$$\begin{aligned}&\lim_{n\to\infty} a_2^{(n)}=a_2, \qquad \lim_{n\to\infty} b_2^{(n)}=b_2, \qquad \lim_{n\to\infty} c_2^{(n)}=c_2,\\ &\lim_{n\to\infty}\frac{a_1^{(n)}}{n}=a_1, \qquad \lim_{n\to\infty}\frac{b_1^{(n)}}{n}=b_1, \qquad \lim_{n\to\infty}\frac{a_0^{(n)}}{n^2}=1,\end{aligned} \tag{2.1}$$

and

$$(b_1a_1-2b_2)^2+(b_1^2-4c_2)(4a_2-a_1^2)>0, \qquad b_1>-2. \tag{2.2}$$

If for every $n\in\mathbb{N}$ the differential equation

$$(a_2^{(n)}+b_2^{(n)}x+c_2^{(n)}x^2)\,y''+(a_1^{(n)}+b_1^{(n)}x)\,y'+a_0^{(n)}y=0 \tag{2.3}$$

possesses a polynomial solution, R_n say, with precise degree n and real zeros only, then

$$c_2+b_1+1=0, \tag{2.4}$$

and for the counting function N_n of the zeros of R_n we have

$$\lim_{n\to\infty}\frac{1}{n}N_n(\xi)=\int_{-\infty}^{\xi}g(a_1,b_1,a_2,b_2;t)\,dt, \tag{2.5}$$

where

$$g(a_1,b_1,a_2,b_2;t):=\frac{1}{2\pi}\frac{\sqrt{4a_2-a_1^2-2t(a_1b_1-2b_2)-(b_1+2)^2t^2}}{a_2+b_2t+c_2t^2}$$

for t being such that the real square root is defined and zero elsewhere.

Proof. First we note that from the assumption on the polynomial solution R_n we have $c_2^{(n)}n(n-1)+b_1^{(n)}n+a_0^{(n)}=0$ for every $n\in\mathbb{N}$, which implies (2.4) by (2.1). Next, we consider the sequence of logarithmic derivatives

$$h_n(z):=\frac{1}{n}\frac{R_n'(z)}{R_n(z)}=\frac{1}{n}\sum_{\nu=1}^{n}\frac{1}{z-x_{n\nu}}, \qquad z\in\mathbb{C}\backslash\mathbb{R}, \tag{2.6}$$

where $x_{n\nu}$ denotes the zeros of R_n, all of them being real by assumption. Substituting $y=R_n$ in (2.3) for h_n we obtain

$$(a_2^{(n)}+b_2^{(n)}z+c_2^{(n)}z^2)\left(\frac{1}{n}h_n'(z)+h_n^2(z)\right)+\left(\frac{a_1^{(n)}}{n}+\frac{b_1^{(n)}}{n}z\right)h_n(z)+\frac{a_0^{(n)}}{n^2}=0 \tag{2.7}$$

for $z\in\mathbb{C}\backslash\mathbb{R}$. Since all the zeros of R_n are real, (2.6) yields $|h_n(z)|\leqslant 1/|\text{Im}\,z|$ for $z\in\mathbb{C}\backslash\mathbb{R}$ which implies that the sequence $(h_n(z))$ is uniformly bounded on every compact subset of $\mathbb{C}\backslash\mathbb{R}$; that is, (h_n) is a normal family on the upper and the lower half planes individually. Further, by Montel's theorem, there is a subsequence converging compactly to some holomorphic function h on $\mathbb{C}\backslash\mathbb{R}$. Since $(h_n'(z))$ again is a a normal family [14, p. 247], from (2.7) we infer

$$(a_2+b_2z+c_2z^2)\,h(z)^2+(a_1+b_1z)\,h(z)+1=0, \qquad z\in\mathbb{C}\backslash\mathbb{R}. \tag{2.8}$$

Putting

$$A(z) := a_2 + b_2 z + c_2 z^2$$

and

$$D(z) := (a_1 + b_1 z)^2 - 4A(z) = (b_1 + 2)^2 z^2 + 2(a_1 b_1 - 2b_2) z + a_1^2 - 4a_2$$

(observe (2.4)), by (2.2), the quadratic equation (2.8) has two different solutions

$$h_\pm(z) = \frac{-(a_1 + b_1 z) \pm \sqrt{D(z)}}{2A(z)} = \frac{2}{-(a_1 + b_1 z) \mp \sqrt{D(z)}} \tag{2.9}$$

with a proper determination of the square root functions. In particular, we mention that, by (2.2), A does not vanish identically and that D has two distinct real roots, r and s with $r < s$, say. Thus each of the functions $h_\pm$ defines an analytic function on the cut plane

$$\mathbb{C}^* := \mathbb{C} \backslash [r, s],$$

and an elementary discussion shows that for $x \in (r, s)$, $A(x)$ is positive and $D(x)$ is negative. Further, by (2.6), for every $n \in \mathbb{N}$ we have

$$0 < \operatorname{Re}(iy\, h_n(iy)) \leqslant 1, \qquad y \in \mathbb{R} \backslash \{0\}, \tag{2.10}$$

giving

$$0 \leqslant \operatorname{Re}(iy\, h(iy)) \leqslant 1, \qquad y \in \mathbb{R}. \tag{2.11}$$

Next, by the use of (2.9), a straightforward computation shows that for one branch, h_- say, we have

$$\lim_{y \to \infty} \operatorname{Re}(iy\, h_-(iy)) = 1,$$

whereas for the other branch h_+ the quantity $\lim_{y \to \infty} \operatorname{Re}(iy\, h_+(iy))$ is not contained in $[0, 1]$. Hence in view of (2.11) we have to make the choice $h = h_-$. Moreover, from (2.6) we obtain

$$\operatorname{Im} h(z) \leqslant 0 \qquad \text{for} \quad \operatorname{Im} z > 0.$$

And equivalently by reflection,

$$\operatorname{Im} h(z) \geqslant 0 \qquad \text{for} \quad \operatorname{Im} z < 0. \tag{2.12}$$

In order to prove the convergence of the whole sequence (h_n), we note that by (2.7) we can decompose (h_n) in two complementary subsequences,

(h_{n_k}) and (h_{m_k}) say, such that $h_{n_k} \to h_-$ and $h_{m_k} \to h_+$ compactly on $\mathbb{C}^*$. However, in view of (2.10) the latter case cannot occur. Finally, using (2.9), (2.12), and [33, pp. 104–105; 31, p. 175], we get

$$g(a_1, b_1, a_2, b_2; t) = \frac{1}{\pi} \operatorname{Im} h(t - i0), \qquad r < t < s,$$

by the inversion formula for the Stieltjes transform [cf. 32, p. 340; 24, p. 188], thereby establishing (2.5). Here we have used that in view of the analytic properties of h the limit distribution possesses a density. (Compare also, e.g., [6, Sects. 2, 3].) ∎

As a first application of the preceding Lemma 3 we consider Jacobi polynomials $P_n^{(\alpha_n, \beta_n)}$ with parameters α_n, β_n depending on the degree n and satisfying $\alpha_n, \beta_n > -1$ for all $n \in \mathbb{N}$. Thus all zeros of $P_n^{(\alpha_n, \beta_n)}$ are located in the interval $[-1, 1]$. Suppose that for

$$A_n := \frac{\alpha_n}{2n + \alpha_n + \beta_n}, \quad B_n := \frac{\beta_n}{2n + \alpha_n + \beta_n}, \qquad n \in \mathbb{N}, \tag{2.13}$$

there exist

$$\lim_{n \to \infty} A_n =: A, \qquad \lim_{n \to \infty} B_n =: B \tag{2.14}$$

and that

$$D := (1 - A + B)(1 + A - B)(1 - A - B)(1 + A + B),$$
$$r := B^2 - A^2 - \sqrt{D}, \qquad s := B^2 - A^2 + \sqrt{D}.$$

Clearly we have $0 \leqslant A,\ B,\ A + B,\ D \leqslant 1$. Numbering the zeros $x_{n\nu}$ of $P_n^{(\alpha_n, \beta_n)}$ according to (0.2) in [22] it was proved that

$$\lim_{n \to \infty} x_{n1} = r, \qquad \lim_{n \to \infty} x_{nn} = s$$

and the zeros are dense in $[r, s]$. If the counting function N_n of the zeros of $P_n^{(\alpha_n, \beta_n)}$ is defined by (0.3), then we prove

THEOREM 2 ($P_n^{(\alpha_n, \beta_n)}$). *Suppose that the notations and assumptions above are satisfied.*

(i) *If $A + B < 1$, then*

$$\lim_{n \to \infty} \frac{1}{n} N_n(\xi) = \frac{1}{\pi(1 - A - B)} \int_r^\xi \frac{\sqrt{(t - r)(s - t)}}{1 - t^2}\, dt, \qquad r < \xi < s. \tag{2.15}$$

(ii) *Assume that* $A+B=1$ *and*

$$D_n := (1-A_n+B_n)(1+A_n-B_n)(1-A_n-B_n)(1+A_n+B_n).$$

(α) *If* $A \neq 1$ *and* $B \neq 1$, *then*

$$\lim_{n\to\infty} \frac{1}{n} N_n(\sqrt{D_n}\,\xi + B_n - A_n) = \frac{2}{\pi}\int_{-1}^{\xi} \sqrt{1-t^2}\,dt, \qquad -1 \leqslant \xi \leqslant 1. \tag{2.16}$$

(β) *If* $A=1$ *and* $\lim_{n\to\infty}(\beta_n/n) = b \in [0,\infty]$, *then*

$$\lim_{n\to\infty} \frac{1}{n} N_n(\sqrt{D_n}\,\xi + B_n - A_n)$$
$$= \frac{2}{\pi}\int_{r_-}^{\xi} \frac{\sqrt{1-1/(4(1+b)) + (1/\sqrt{1+b})\,t - t^2}}{1+(2/\sqrt{1+b})\,t}\,dt, \tag{2.17}$$

$r_- \leqslant \xi \leqslant r_+$, *where* $r_\pm = 1/(2\sqrt{1+b}) \pm 1$.

(γ) *If* $B=1$ *and* $\lim_{n\to\infty}(\alpha_n/n) = a \in [0,\infty]$, *then*

$$\lim_{n\to\infty} \frac{1}{n} N_n(\sqrt{D_n}\,\xi + B_n - A_n)$$
$$= \frac{2}{\pi}\int_{s_-}^{\xi} \frac{\sqrt{1-1/(4(1+a)) - (1/\sqrt{1+a})\,t - t^2}}{1-(2/\sqrt{1+a})\,t}\,dt, \tag{2.18}$$

$s_- \leqslant \xi \leqslant s_+$, *where* $s_\pm = -1/(2\sqrt{1+a}) \pm 1$.

Proof. In view of Lemma 3 we put $y(x) := P_n^{(\alpha_n,\beta_n)}(\gamma_n x + \delta_n)$ where $\gamma_n > 0$, $\delta_n \in \mathbb{R}$ have to be determined suitably. Then y satisfies the differential equation (2.3) (e.g., use (4.2.1) in [29, p. 60]), where

$$\begin{aligned}
&a_2^{(n)} := \frac{1-\delta_n^2}{\gamma_n^2}\,\frac{n}{n+\alpha_n+\beta_n+1}, \qquad b_2^{(n)} := -\frac{2\delta_n}{\gamma_n}\,\frac{n}{n+\alpha_n+\beta_n+1},\\
&c_2^{(n)} := -\frac{n}{n+\alpha_n+\beta_n+1},\\
&a_1^{(n)} := \frac{1}{\gamma_n}\,(\beta_n - \alpha_n - (\alpha_n+\beta_n+2)\,\delta_n)\,\frac{n}{n+\alpha_n+\beta_n+1},\\
&b_1^{(n)} := -\frac{\alpha_n+\beta_n+2}{n+\alpha_n+\beta_n+1}\,n, \qquad \alpha_0^{(n)} := n^2.
\end{aligned} \tag{2.19}$$

(i) If $A+B<1$, then we put $\gamma_n:=1$ and $\delta_n:=0$, and in view of (2.13), (2.14) the limits in (2.1) are given by

$$a_2=\frac{1-A-B}{1+A+B}=-c_2, \qquad b_2=0, \qquad a_1=\frac{2(B-A)}{1+A+B},$$
$$b_1=-\frac{2(A+B)}{1+A+B},$$

and they satisfy (2.2). Hence (2.15) follows from Lemma 3.

(ii) If $A+B=1$, then we put $\gamma_n:=\sqrt{D_n}$, $\delta_n:=B_n-A_n$. Now the quantities in (2.1) are given by

(α) $a_2=\frac{1}{4}$, $b_2=c_2=a_1=0$, $b_1=-1$;

(β) $a_2=\frac{1}{4}$, $b_2=1/(2\sqrt{1+b})=-a_1$, $c_2=0$, $b_1=-1$;

(γ) $a_2=\frac{1}{4}$, $b_2=-1/(2\sqrt{1+a})=-a_1$, $c_2=0$, $b_1=-1$

in the cases α, β, and γ respectively, and they fulfill (2.2). Again Lemma 3 completes the proof. ∎

Remarks. (i) Note that $A+B<1$ is equivalent to the existence of both limits

$$\lim_{n\to\infty}\frac{\alpha_n}{n}=:a=\frac{2A}{1-A-B}, \qquad \lim_{n\to\infty}\frac{\beta_n}{n}=:b=\frac{2B}{1-A-B},$$

whereas $A+B=1$ is the same as $\lim_{n\to\infty}(\alpha_n/n)=\infty$ or $\lim_{n\to\infty}(\beta_n/n)=\infty$. In the latter case we have

$$\sqrt{D_n}=\frac{4\sqrt{n(n+\alpha_n)(n+\beta_n)(n+\alpha_n+\beta_n)}}{(2n+\alpha_n+\beta_n)^2}.$$

(ii) Other choices of γ_n and δ_n in the Proof of Theorem 2 would produce other limit distributions within the class given by Lemma 3 (cf. [4]). We have restricted our considerations to the cases stated in Theorem 2, since they cover all important choices of α_n and β_n satisfying (2.14). If $A+B=1$, then, by (2.19), we always get $c_2=0$ in (2.5). In particular, for (2.17) we mention the cases

$$\lim_{n\to\infty}\frac{1}{n}N_n(\sqrt{D_n}\,\xi+B_n-A_n)=\frac{2}{\pi}\int_{-1}^{\xi}\sqrt{1-t^2}\,dt, \qquad -1\leqslant\xi\leqslant1,$$

and

$$\lim_{n\to\infty}\frac{1}{n}N_n(\sqrt{D_n}\,\xi+B_n-A_n)=\frac{1}{\pi}\int_{-1/2}^{\xi}\sqrt{\frac{3-2t}{1+2t}}\,dt, \qquad -\frac{1}{2}\leqslant\xi\leqslant\frac{3}{2},$$

if $b=\infty$ and $b=0$ respectively. As an example for a different choice of γ_n and δ_n we exhibit the following one.

Let $\alpha_n/n \to \infty$ and $\beta_n/n \to b$ as $n \to \infty$, then in (2.19) choose $\delta_n := -1$ and $\gamma_n := 2n/\alpha_n$. Lemma 3 implies

$$\lim_{n\to\infty} \frac{1}{n} N_n\left(\frac{2n}{\alpha_n}\xi - 1\right) = \frac{1}{2\pi}\int_{\tau_1}^{\xi} \sqrt{(t-\tau_1)(\tau_2 - t)}\,\frac{dt}{t}, \qquad \tau_1 < \xi < \tau_2,$$

where $\tau_{1,2} = b + 2 \pm 2\sqrt{b+1}$.

(iii) In view of the normalization in (2.16)–(2.18) we mention that the centroid $B_n - A_n = (\beta_n - \alpha_n)/(2n + \alpha_n + \beta_n) = (1/n)\sum_{\nu=1}^{n} x_{n\nu}$ can be regarded as the mean value of the zeros of the nth polynomial (cf. [5]).

Finally, we give the corresponding limit distributions for the Laguerre polynomials $L_n^{(\alpha_n)}$ and the Hermite polynomials $H_n^{(\alpha_n)}$ which in the case $\alpha_n/n \to \infty$ have been found recently by Dette and Studden [4] using analytic continued fractions. For completeness we state the results which can be derived from Lemma 3 again as Theorem 2 above. Therefore we omit the straightforward proofs.

THEOREM 3 ($L_n^{(\alpha_n)}$). *Suppose that* $\alpha_n > -1$.

(i) *If*

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = a \qquad \textit{and} \qquad \kappa_n := n + \frac{\alpha_n + 1}{2}, \tag{2.20}$$

then

$$\lim_{n\to\infty} \frac{1}{n} N_n(\kappa_n \xi) = \frac{2+a}{4\pi}\int_r^{\xi} \sqrt{(t-r)(s-t)}\,\frac{dt}{t}, \qquad r < \xi < s, \tag{2.21}$$

where $r := 2 - (4/(2+a))\sqrt{a+1}$, $s := 2 + (4/(2+a))\sqrt{a+1}$.

(ii) *If* $\lim_{n\to\infty}(\alpha_n/n) = \infty$, *then*

$$\lim_{n\to\infty} \frac{1}{n} N_n(2\sqrt{n\alpha_n}\,\xi + \alpha_n) = \frac{2}{\pi}\int_{-1}^{\xi} \sqrt{1-t^2}\,dt, \qquad -1 \leqslant \xi \leqslant 1.$$

The essential property of the scaling sequence κ_n in (2.20) and (2.21) is that it grows like a positive multiple of n. The particular choice in (2.20) is made in the light of known formulae for Laguerre polynomials and in order to have an easy comparison with various special cases and related asymptotics [cf. 29, p. 200; 7]. The corresponding limit distributions for the Hermite polynomials $H_n^{(\alpha_n)}$ are readily deduced from Theorem 3 by using the connection formulae [see 2, p. 156; 6, pp. 52, 53, 60]

$$H_{2k}^{(\alpha)}(z) = (-1)^k\, 2^{2k} k!\, L_k^{(\alpha-(1/2))}(z^2),$$
$$H_{2k+1}^{(\alpha)}(z) = (-1)^k\, 2^{2k+1} k!\, z L_k^{(\alpha+(1/2))}(z^2)$$

for $k \in \mathbb{N}_0$.

THEOREM 4 ($H_n^{(\alpha_n)}$). *Suppose that* $\alpha_n > -1/2$.

(i) *If*

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = a \qquad \textit{and} \qquad \lambda_n := 2n + 2\alpha_n + 1,$$

then

$$\lim_{n\to\infty} \frac{1}{n} N_n(\sqrt{\lambda_n}\,\xi) = \int_{-\sigma}^{\xi} g_a(t)\,dt, \qquad -\sigma < \xi < \sigma,$$

where

$$g_a(t) := \begin{cases} \dfrac{2(1+a)}{\pi} \dfrac{\sqrt{(t^2-\rho^2)(\sigma^2-t^2)}}{|t|}, & \rho < |t| < \sigma \\ 0, & \textit{elsewhere} \end{cases}$$

and

$$\rho := \left(\frac{1}{2}\left(1 - \frac{\sqrt{1+2a}}{1+a}\right)\right)^{1/2}, \qquad \sigma := \left(\frac{1}{2}\left(1 + \frac{\sqrt{1+2a}}{1+a}\right)\right)^{1/2}.$$

(ii) *If* $\lim_{n\to\infty}(\alpha_n/n) = \infty$, *then*

$$\lim_{n\to\infty} \frac{1}{n} N_n\left(\sqrt{\sqrt{2n\alpha_n}\,\xi + \alpha_n}\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\xi} \sqrt{1-t^2}\,dt, \qquad 0 \leqslant \xi \leqslant 1,$$

and

$$\lim_{n\to\infty} \frac{1}{n} N_n\left(-\sqrt{\sqrt{2n\alpha_n}\,|\xi| + \alpha_n}\right) = \frac{1}{2} - \frac{1}{\pi}\int_0^{|\xi|} \sqrt{1-t^2}\,dt, \qquad -1 \leqslant \xi \leqslant 0.$$

REFERENCES

1. P. BILLINGSLEY, "Probability and Measure," Wiley, New York, 1979.
2. T. CHIHARA, "An Introduction to Orthogonal Polynomials," Gordon & Breach, New York, 1978.

3. H. Dette and W. Studden, On a new characterization of the classical orthogonal polynomials, *J. Approx. Theory* **71** (1992), 3–17.
4. H. Dette and W. Studden, Some new asymptotic properties for the zeros of Jacobi, Laguerre, and Hermite polynomials, *Constructive Approx.* **11** (1995), to appear.
5. A. Elbert and A. Laforgia, New properties of the zeros of a Jacobi polynomial in relation to their centroid, *SIAM J. Math. Anal.* **18** (1987), 1563–1572.
6. W. Gawronski, On the asymptotic distribution of Hermite, Laguerre, and Jonquière polynomials, *J. Approx. Theory* **50** (1987), 214–231.
7. W. Gawronski, Strong asymptotics and the asymptotic zero distributions of Laguerre polynomials $L_n^{(an+\alpha)}$ and Hermite polynomials $H_n^{(an+\alpha)}$, *Analysis* **13** (1993), 29–67.
8. W. Gawronski and B. Shawyer, Strong asymptotics and the limit distributions of the zeros of Jacobi polynomials $P_n^{(an+\alpha, bn+\beta)}$, *in* "Progress in Approximation Theory" (P. Nevai and A. Pinkus, Eds.), pp. 379–404, Academic Press, New York, 1991.
9. W. Gawronski and U. Stadtmüller, On the zeros of power series with exponential logarithmic coefficients, *Canad. Math. Bull.* **24** (1981), 257–271.
10. W. Gawronski and U. Stadtmüller, On the zeros of Jonquière's function with a large complex parameter, *Michigan Math. J.* **31** (1984), 275–293.
11. W. Gawronski and U. Stadtmüller, On the zeros of Lerch's transcendental function with real parameters, *J. Approx. Theory* **53** (1988), 354–364.
12. A. Gonchar and E. Rakhmanov, On convergence of simultaneous Padé approximants for systems of Markov type, *Proc. Steklov Inst. Math.* **157** (1983), 31–50.
13. A. Gonchar and E. Rakhmanov, Equilibrium measure and the distribution of zeros of extremal polynomials, *Math. USSR Sb.* **53** (1986), 119–130.
14. E. Hille, "Analytic Function Theory," Vol. 2, Chelsea, New York, 1973.
15. W. Jurkat and A. Peyerimhoff, On power series with negative zeros, *Tôhoku Math. J.* **24** (1972), 207–221.
16. D. Lawden, The function $\sum_1^\infty n^r z^n$ and associated polynomials, *Proc. Cambridge Philos. Soc.* **47** (1951), 309–314.
17. E. Lindelöf, "Le calcul des résidus et ses applications à la théorie des fonctions," Gauthier-Villars, Paris, 1905.
18. G. Meinardus and G. Merz, Zur periodischen Spline-Interpolation, *in* "Spline Funktionen" (K. Böhmer, G. Meinardus, and W. Schempp, Eds.), pp. 177–195, Bibliographisches Institut, Mannheim, 1974.
19. H. Mhaskar and E. Saff, Polynomials with Laguerre weights in L^p, *in* "Rational Approximation and Interpolation" (P. Graves-Morris, E. Saff, and R. Varga, Eds.), Lecture Notes in Mathematics, Vol. 1105, pp. 511–523, Springer-Verlag, New York/Berlin, 1984.
20. M. Mhaskar and E. Saff, Weighted polynomials on finite and infinite intervals: A unified approach, *Bull. Amer. Math. Soc.* **11** (1984), 351–354.
21. H. Mhaskar and E. Saff, Where does the sup-norm of a weighted polynomial live? (A generalization of incomplete polynomials), *Constructive Approx.* **1** (1985), 71–91.
22. D. Moak, E. Saff, and R. Varga, On the zeros of Jacobi polynomials $P_n^{(\alpha_n, \beta_n)}(x)$, *Trans. Amer. Math. Soc.* **11** (1984), 351–354.
23. H. ter Morsche, On the existence and convergence of interpolating periodic spline functions, *in* "Spline Funktionen" (K. Böhmer, G. Meinardus, and W. Schempp, Eds.), pp. 197–214, Bibliographisches Institut, Mannheim, 1974.
24. O. Perron, "Die Lehre von den Kettenbrüchen," Vol. 2, Teubner, Stuttgart, 1957.
25. A. Peyerimhoff, "Lectures on Summability," Lecture Notes in Mathematics, Vol. 107, Springer-Verlag, Berlin, 1969.
26. A. Peyerimhoff, "Gewöhnliche Differentialgleichungen II," Akademische Verlagsgesellschaft, Wiesbaden, 1982.

27. G. Polya and G. Szegö, "Aufgaben und Lehrsätze aus der Analysis I," Springer-Verlag, Berlin, 1970.
28. M. Reimer, The main-roots of the Euler–Frobenius polynomials, *J. Approx. Theory* **45** (1985), 358–363.
29. G. Szegö, "Orthogonal Polynomials," *Amer. Math. Soc. Colloq. Publ.*, Vol. 23, Providence, RI, 1975.
30. C. Truesdell, On a function which occurs in the theory of the structure of polymers, *Ann. of Math.* **46** (1945), 144–157.
31. W. Van Assche, "Asymptotics for Orthogonal Polynomials," Lecture Notes in Mathematics, Vol. 1265, Springer-Verlag, Berlin/Heidelberg/New York, 1987.
32. D. Widder, "The Laplace Transform," Princeton Univ. Press, Princeton, NJ, 1966.
33. A. Wintner, "Spektraltheorie der unendlichen Matrizen," Hirzel, Leipzig, 1929.
34. E. Wirsing, On the monotonicity of the zeros of two power series, *Michigan Math. J.* **13** (1966), 215–218.
35. W. Wirtinger, Über eine besondere Dirichletsche Reihe, *J. Reine Angew. Math.* **129** (1905), 214–219.